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Research Article

Radius Constants for Analytic Functions with Fixed Second Coefficient

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Let $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ be analytic in the unit disk with the second coefficient a_2 satisfying $|a_2|=2b$, $0\le b\le 1$. Sharp radius of Janowski starlikeness is obtained for functions f whose nth coefficient satisfies $|a_n|\le cn+d$ $(c,d\ge 0)$ or $|a_n|\le c/n$ $(c>0$ and $n\ge 3)$. Other radius constants are also obtained for these functions, and connections with earlier results are made.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the open unit disk $\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\}$, normalized by $f(0)=0=f'(0)-1$, and let $\mathcal{S}$ denote its subclass consisting of univalent functions. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in\mathcal{S}$, de Branges [1] obtained the sharp coefficient bound that $|a_n|\le n$ $(n\ge 2)$. However, the inequality $|a_n|\le n$, $n\ge 2$, is not sufficient for f to be univalent; for example, $f(z)=z+2z^2$ is clearly not a member of $\mathcal{S}$.

Several subclasses of $\mathcal{S}$ possess a similar coefficient bound. For instance, the nth coefficients of starlike functions, convex functions in the direction of imaginary axis, and close-to-convex functions satisfy $|a_n|\le n$ $(n\ge 2)$ [2–4]. Other examples include functions which are convex, starlike of order 1/2, and starlike with respect to symmetric points. The nth coefficients of these functions satisfy $|a_n|\le 1$ $(n\ge 2)$ [5–7]. The nth coefficient of close-to-convex functions with argument β satisfies $|a_n|\le 1+(n-1)\cos\beta$ [8], and the coefficients of uniformly starlike functions are bounded by $2/n$ [9], while $|a_n|\le 1/n$ [10] for uniformly convex functions. Simple examples show that these bounds are not sufficient to characterize the geometric properties of the classes of functions.

In the sequel, we will assume that $f\in\mathcal{A}$ has the Taylor expansion of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. Gavrilov [11] showed that the radius of univalence for functions $f\in\mathcal{A}$ satisfying $|a_n|\le n$ $(n\ge 2)$ is the real root $r_0\simeq 0.164$ of the equation $2(1-r)^3-(1+r)=0$, and the result is sharp for $f(z)=2z-z/(1-z)^2$. Gavrilov also proved that the radius of univalence for functions $f\in\mathcal{A}$ satisfying the coefficient bound $|a_n|\le M$ $(n\ge 2)$ is $1-\sqrt{M/(1+M)}$. The condition $|a_n|\le M$ clearly holds for functions $f\in\mathcal{A}$ satisfying $|f(z)|\le M$, and for these functions, Landau [12] proved that the radius of univalence is $M-\sqrt{M^2-1}$. In fact, Yamashita [13] showed that the radius of univalence obtained by Gavrilov [11] is also the radius of starlikeness for functions $f\in\mathcal{A}$ satisfying $|a_n|\le n$ or $|a_n|\le M$. Additionally, Yamashita [13] determined that the radius of convexity for functions $f\in\mathcal{A}$ satisfying $|a_n|\le n$ is the real root $r_0\simeq 0.090$ of the equation $2(1-r)^4-(1+4r+r^2)=0$, while the radius of convexity for functions $f\in\mathcal{A}$ satisfying $|a_n|\le M$ is the real root of

$$(M+1)(1-r)^3-M(1+r)=0. \tag{1}$$

Recently, Kalaj et al. [14] obtained the radii of univalence, starlikeness, and convexity for harmonic mappings satisfying certain coefficient inequalities.

For two analytic functions f and g, the function f is *subordinate* to g, denoted by $f \prec g$, if there is an analytic self-map w of $\mathbb{D}$ with $w(0) = 0$ satisfying $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

For $\beta \in \mathbb{R} \setminus \{1\}$, $\alpha \geq 0$, the class $\mathscr{L}(\alpha, \beta)$ consists of functions $f \in \mathscr{A}$ satisfying

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\beta) z}{1 - z}. \tag{2}$$

Denote by $\mathscr{L}_0(\alpha, \beta)$ its subclass consisting of functions $f \in \mathscr{A}$ satisfying

$$\left|\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} - 1\right| \leq |1 - \beta| \quad (\beta \in \mathbb{R} \setminus \{1\}, \alpha \geq 0). \tag{3}$$

These classes were investigated in [15–24].

For $\beta < 1$, the class $\mathscr{L}(0, \beta)$ is the class of starlike functions of order β, while, for the case $\beta > 1$, the class was studied in [25–28].

The class $\mathcal{ST}[A, B]$ of Janowski starlike functions [29] consists of $f \in \mathscr{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1). \tag{4}$$

Certain well-known subclasses of starlike functions are special cases of $\mathcal{ST}[A, B]$ for appropriate choices of the parameters A and B. For example, for $0 \leq \beta < 1$, $\mathcal{ST}(\beta) := \mathcal{ST}[1 - 2\beta, -1]$ is the familiar class of starlike functions of order β. Denote by $\mathcal{ST}_\beta$ the class $\mathcal{ST}_\beta := \mathscr{L}_0(0, \beta) = \mathcal{ST}[1 - \beta, 0]$. Janowski [29] obtained the sharp radius of convexity for $\mathcal{ST}[A, B]$.

This paper studies the class $\mathscr{A}_b$ consisting of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $(|a_2| = 2b,\ 0 \leq b \leq 1)$, in the disk $\mathbb{D}$. The subclass of univalent functions in $\mathscr{A}_b$ have been studied in [30–33]. In [33], Ravichandran obtained sharp radii of starlikeness and convexity of order α for functions $f \in \mathscr{A}_b$ satisfying $|a_n| \leq n$ or $|a_n| \leq M$, $n \geq 3$. The author also obtained the radius of uniform convexity and parabolic starlikeness for functions $f \in \mathscr{A}_b$ satisfying $|a_n| \leq n$, $n \geq 3$.

This paper finds radius constants for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}_b$ satisfying either $|a_n| \leq cn + d$ $(c, d \geq 0)$ or $|a_n| \leq c/n$ $(c > 0, n \geq 3)$. In the next section, sharp $\mathscr{L}(\alpha, \beta)$-radius and $\mathcal{ST}[A, B]$-radius are derived for these classes. Several known radius constants are shown to be special cases of the results obtained.

2. Radius Constants

A sufficient condition for functions $f \in \mathscr{A}$ to belong to the class $\mathscr{L}(\alpha, \beta)$ is given in the following lemma.

Lemma 1 (see [24, 34]). *Let $\beta \in \mathbb{R} \setminus \{1\}$ and $\alpha \geq 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1 - \alpha) n - \beta\right) |a_n| \leq |1 - \beta|, \tag{5}$$

then $f \in \mathscr{L}(\alpha, \beta)$.

Making use of this lemma, the sharp $\mathscr{L}(\alpha, \beta)$-radius is obtained for $f \in \mathscr{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$.

Theorem 2. *Let $\beta \in \mathbb{R} \setminus \{1\}$, $6\alpha + 3 - \beta \geq 0$, and $\alpha \geq 0$. The $\mathscr{L}(\alpha, \beta)$-radius for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$, $c, d \geq 0$, $n \geq 3$, is the real root in $(0, 1)$ of the equation*

$$\begin{aligned} &((c + d)(1 - \beta) + |1 - \beta| \\ &\quad + (2\alpha + 2 - \beta)(2(c - b) + d) r)(1 - r)^4 \\ &\quad = c\alpha\left(1 + 4r + r^2\right) + ((1 - \alpha) c + \alpha d)\left(1 - r^2\right) \\ &\qquad + ((1 - \alpha) d - \beta c)(1 - r)^2 - \beta d(1 - r)^3. \end{aligned} \tag{6}$$

For $\beta < 1$, this number is also the $\mathscr{L}_0(\alpha, \beta)$-radius of $f \in \mathscr{A}_b$. The results are sharp.

Proof. The number r_0 is the $\mathscr{L}(\alpha, \beta)$-radius for $f \in \mathscr{A}_b$ if and only if $f(r_0 z)/r_0 \in \mathscr{L}(\alpha, \beta)$. Therefore, by Lemma 1, it is sufficient to verify the inequality

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1 - \alpha) n - \beta\right) |a_n| r_0^{n-1} \leq |1 - \beta|, \tag{7}$$

where r_0 is the real root in $(0, 1)$ of (6). Using the known expansions

$$\sum_{n=3}^{\infty} r_0^{n-1} = \frac{1}{1 - r_0} - 1 - r_0, \tag{8}$$

$$\sum_{n=3}^{\infty} n r_0^{n-1} = \frac{1}{(1 - r_0)^2} - 1 - 2r_0, \tag{9}$$

$$\sum_{n=3}^{\infty} n^2 r_0^{n-1} = \frac{1 + r_0}{(1 - r_0)^3} - 1 - 4r_0, \tag{10}$$

$$\sum_{n=3}^{\infty} n^3 r_0^{n-1} = \frac{1 + 4r_0 + r_0^2}{(1 - r_0)^4} - 1 - 8r_0 \tag{11}$$

leads to

$$\begin{aligned} &\sum_{n=2}^{\infty} \left(\alpha n^2 + (1 - \alpha) n - \beta\right) |a_n| r_0^{n-1} \\ &\quad \leq 2(2\alpha + 2 - \beta) b r_0 \\ &\qquad + \sum_{n=3}^{\infty} \left(\alpha n^2 + (1 - \alpha) n - \beta\right)(cn + d) r_0^{n-1} \\ &\quad = 2(2\alpha + 2 - \beta) b r_0 + c\alpha\left(\frac{1 + 4r_0 + r_0^2}{(1 - r_0)^4} - 1 - 8r_0\right) \\ &\qquad + ((1 - \alpha) c + \alpha d)\left(\frac{1 + r_0}{(1 - r_0)^3} - 1 - 4r_0\right) \\ &\qquad + ((1 - \alpha) d - \beta c)\left(\frac{1}{(1 - r_0)^2} - 1 - 2r_0\right) \\ &\qquad - \beta d\left(\frac{1}{1 - r_0} - 1 - r_0\right) \end{aligned}$$

$$
\begin{aligned}
&= (c+d)(\beta-1) - (2\alpha+2-\beta)(2(c-b)+d)r_0 \\
&\quad + \left(c\alpha\left(1+4r_0+r_0^2\right) + ((1-\alpha)c+\alpha d)\left(1-r_0^2\right)\right. \\
&\qquad + ((1-\alpha)d-\beta c)(1-r_0)^2 \\
&\qquad \left. - \beta d(1-r_0)^3\right) \times (1-r_0)^{-4} \\
&= |1-\beta|.
\end{aligned} \tag{12}
$$

For $\beta < 1$, consider the function

$$
\begin{aligned}
f_0(z) &= z - 2bz^2 - \sum_{n=3}^{\infty}(cn+d)z^n \\
&= (c+1)z + 2(c-b)z^2 - \frac{cz}{(1-z)^2} - \frac{dz^3}{1-z}.
\end{aligned} \tag{13}
$$

At the root $z = r_0$ in $(0,1)$ of (6), f_0 satisfies

$$
\operatorname{Re}\left(\alpha\frac{z^2 f''_0(z)}{f_0(z)} + \frac{zf'_0(z)}{f_0(z)}\right) = 1 - \frac{N(r_0)}{D(r_0)} = \beta, \tag{14}
$$

where

$$
\begin{aligned}
N(r_0) &= -2(c-b)(2\alpha+1)r_0 + \frac{2cr_0(2\alpha+1)}{(1-r_0)^3} \\
&\quad + \frac{6c\alpha r_0^2}{(1-r_0)^4} + \frac{2dr_0^2(3\alpha+1)}{1-r_0} \\
&\quad + \frac{dr_0^3(6\alpha+1)}{(1-r_0)^2} + \frac{2dr_0^4\alpha}{(1-r_0)^3},
\end{aligned} \tag{15}
$$

$$
D(r_0) = c + 1 + 2(c-b)r_0 - \frac{c}{(1-r_0)^2} - \frac{dr_0^2}{1-r_0}.
$$

This shows that r_0 is the sharp $\mathscr{L}(\alpha,\beta)$-radius for $f \in \mathscr{A}_b$. For $\beta < 1$, (14) shows that the rational expression $N(r_0)/D(r_0)$ is positive, and therefore the equality

$$
\left|\alpha\frac{z^2 f''_0(z)}{f_0(z)} + \frac{zf'_0(z)}{f_0(z)} - 1\right| = 1 - \beta \tag{16}
$$

holds. Thus, r_0 is the sharp $\mathscr{L}_0(\alpha,\beta)$-radius for $f \in \mathscr{A}_b$ when $\beta < 1$.

For $\beta > 1$, the function

$$
\begin{aligned}
f_0(z) &= z + 2bz^2 + \sum_{n=3}^{\infty}(cn+d)z^n \\
&= (1-c)z + 2(b-c)z^2 + \frac{cz}{(1-z)^2} + \frac{dz^3}{1-z}
\end{aligned} \tag{17}
$$

demonstrates sharpness of the result. The derivation is similar to the case $\beta < 1$ and is omitted. □

Theorem 3. *Let $\beta \in \mathbb{R}\setminus\{1\}$ and $\alpha \ge 0$. The $\mathscr{L}(\alpha,\beta)$-radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}_b$ satisfying the coefficient inequality $|a_n| \le c/n$ for $n \ge 3$ and $c > 0$ is the real root in $(0,1)$ of the equation*

$$
\begin{aligned}
&\left[c(1-\beta) + |1-\beta| + (2\alpha+2-\beta)r\left(\frac{c}{2}-2b\right)\right](1-r)^2 \\
&\quad = c\alpha + (1-\alpha)c(1-r) + \beta c(1-r)^2\frac{\log(1-r)}{r}.
\end{aligned} \tag{18}
$$

For $\beta < 1$, this number is also the $\mathscr{L}_0(\alpha,\beta)$-radius of $f \in \mathscr{A}_b$. The results are sharp.

Proof. By Lemma 1, r_0 is the $\mathscr{L}(\alpha,\beta)$-radius of functions $f \in \mathscr{A}_b$ when inequality (7) holds for the real root r_0 of (18) in $(0,1)$. Using (8) and (9) together with

$$
\sum_{n=3}^{\infty}\frac{r_0^{n-1}}{n} = -\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2} \tag{19}
$$

leads to

$$
\begin{aligned}
&\sum_{n=2}^{\infty}\left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| r_0^{n-1} \\
&\quad \le 2(2\alpha+2-\beta)br_0 \\
&\qquad + \sum_{n=3}^{\infty}\left(\alpha n^2 + (1-\alpha)n - \beta\right)\left(\frac{c}{n}\right)r_0^{n-1} \\
&\quad = 2(2\alpha+2-\beta)br_0 + c\alpha\left(\frac{1}{(1-r_0)^2} - 1 - 2r_0\right) \\
&\qquad + (1-\alpha)c\left(\frac{1}{1-r_0} - 1 - r_0\right) \\
&\qquad - \beta c\left(-\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2}\right) \\
&\quad = c(\beta-1) + (2\alpha+2-\beta)r_0\left(2b - \frac{c}{2}\right) \\
&\qquad + \frac{c\alpha r_0 + (1-\alpha)c(1-r_0)r_0 + \beta c(1-r_0)^2\log(1-r_0)}{(1-r_0)^2 r_0} \\
&\quad = |1-\beta|.
\end{aligned} \tag{20}
$$

To verify sharpness for $\beta < 1$, consider the function

$$
\begin{aligned}
f_0(z) &= z - 2bz^2 - \sum_{n=3}^{\infty}\frac{c}{n}z^n \\
&= (1+c)z + \left(\frac{c}{2}-2b\right)z^2 + c\log(1-z).
\end{aligned} \tag{21}
$$

At the root $z = r_0$ in $(0, 1)$ of (18), f_0 satisfies

$$\begin{aligned}
&\operatorname{Re}\left(\alpha\frac{z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)}\right) \\
&\quad = 1 - \left(-\left(\frac{c}{2} - 2b\right) r_0 (2\alpha + 1) + \frac{c r_0 \alpha}{(1 - r_0)^2}\right. \\
&\qquad \left. + \frac{c}{1 - r_0} + \frac{c \log(1 - r_0)}{r_0}\right) \\
&\qquad \times \left((1 + c) + \left(\frac{c}{2} - 2b\right) r_0 + \frac{c \log(1 - r_0)}{r_0}\right)^{-1} = \beta.
\end{aligned} \tag{22}$$

Thus, r_0 is the sharp $\mathscr{L}(\alpha, \beta)$-radius for $f \in \mathscr{A}_b$. For $\beta < 1$, the rational expression in (22) is positive, and therefore

$$\left|\alpha\frac{z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)} - 1\right| = 1 - \beta, \tag{23}$$

which shows that r_0 is the sharp $\mathscr{L}_0(\alpha, \beta)$-radius for $f \in \mathscr{A}_b$. For $\beta > 1$, sharpness of the result is demonstrated by the function f_0 given by

$$\begin{aligned}
f_0(z) &= z + 2bz^2 + \sum_{n=3}^{\infty} \frac{c}{n} z^n \\
&= (1 - c) z + \left(2b - \frac{c}{2}\right) z^2 - c \log(1 - z).
\end{aligned} \tag{24}$$

□

Remark 4. The results obtained above yield the following special cases.

(1) For $\alpha = 0$, $\beta = 0$, $c = 1$, $d = 0$, and $0 \le b \le 1$, Theorem 2 yields the radius of starlikeness obtained by Yamashita [13].

(2) For $\alpha = 0$, $c = 1$, and $d = 0$, Theorem 2 reduces to Theorem 2.1 in [33, page 3]. When $\alpha = 0$, $c = 0$, and $d = M$, Theorem 2 leads to Theorem 2.5 in [33, page 5].

(3) For $\alpha = 0$, Theorem 3 yields the radius of starlikeness of order β for $f \in \mathscr{A}_b$ obtained by Ravichandran [33, Theorem 2.8].

The following result of Goel and Sohi [35] will be required in our investigation of the class of Janowski starlike functions.

Lemma 5 (see [35]). *Let $-1 \le B < A \le 1$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} ((1 - B) n - (1 - A)) |a_n| \le A - B, \tag{25}$$

then $f \in \mathscr{ST}[A, B]$.

The next result finds the sharp $\mathscr{ST}[A, B]$-radius for $f \in \mathscr{A}_b$ satisfying the coefficient inequality $|a_n| \le cn + d$.

Theorem 6. *Let $-1 \le B < A \le 1$. The $\mathscr{ST}[A, B]$-radius for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{A}_b$ satisfying the coefficient inequality $|a_n| \le cn + d$, $n \ge 3$ and $c, d \ge 0$, is the real root in $(0, 1)$ of the equation*

$$\begin{aligned}
&[(A - B)(c + d + 1) \\
&\quad - (2b - 2c - d)(2(1 - B) - (1 - A)) r](1 - r)^3 \\
&\qquad = c(1 - B)(1 + r) + (d(1 - B) - c(1 - A))(1 - r) \\
&\qquad\quad - (1 - A) d (1 - r)^2.
\end{aligned} \tag{26}$$

This radius is sharp.

Proof. It is evident that r_0 is the $\mathscr{ST}[A, B]$-radius of $f \in \mathscr{A}_b$ if and only if $f(r_0 z)/r_0 \in \mathscr{ST}[A, B]$. Hence, by Lemma 5, it suffices to show that

$$\begin{aligned}
&\sum_{n=2}^{\infty} ((1 - B) n - (1 - A)) |a_n| r_0^{n-1} \le A - B \\
&\qquad (-1 \le B < A \le 1),
\end{aligned} \tag{27}$$

where r_0 is the root in $(0, 1)$ of (26). From (8), (9), and (10), it follows that

$$\begin{aligned}
&\sum_{n=2}^{\infty} ((1 - B) n - (1 - A)) |a_n| r_0^{n-1} \\
&\quad \le 2(2(1 - B) - (1 - A)) b r_0 \\
&\qquad + \sum_{n=3}^{\infty} ((1 - B) n - (1 - A))(cn + d) r_0^{n-1} \\
&\quad = 2(2(1 - B) - (1 - A)) b r_0 \\
&\qquad + c(1 - B)\left(\frac{1 + r_0}{(1 - r_0)^3} - 1 - 4r_0\right) \\
&\qquad + (d(1 - B) - c(1 - A))\left(\frac{1}{(1 - r_0)^2} - 1 - 2r_0\right) \\
&\qquad - (1 - A) d\left(\frac{1}{1 - r_0} - 1 - r_0\right) \\
&\quad = (B - A)(c + d) + (2b - 2c - d) \\
&\qquad \times (2(1 - B) - (1 - A)) r_0 \\
&\qquad + \left(c(1 - B)(1 + r_0)\right. \\
&\qquad\quad + (d(1 - B) - c(1 - A))(1 - r_0) \\
&\qquad\quad \left. - (1 - A) d (1 - r_0)^2\right) \times (1 - r_0)^{-3} \\
&\quad = A - B.
\end{aligned} \tag{28}$$

The function f_0 given by (13) shows that the result is sharp. Indeed, at the point $z = r_0$ where r_0 is the root in $(0, 1)$ of (26), the function f_0 satisfies

$$\begin{aligned}
&\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| \\
&\quad = \left(-2(c-b)r_0 + \frac{2dr_0^2}{1-r_0} + \frac{dr_0^3}{(1-r_0)^2} + \frac{2cr_0}{(1-r_0)^3}\right) \\
&\qquad \times \left(c+1+2(c-b)r_0 - \frac{c}{(1-r_0)^2} - \frac{dr_0^2}{1-r_0}\right)^{-1}, \\
&\left|A - B\frac{zf_0'(z)}{f_0(z)}\right| \\
&\quad = \frac{(c+1)(A-B) + 2(c-b)r_0(A-2B)}{c+1+2(c-b)r_0 - c/(1-r_0)^2 - dr_0^2/(1-r_0)} \\
&\qquad - \left(\frac{c(A-B)}{(1-r_0)^2} + \frac{2cr_0B}{(1-r_0)^3}\right. \\
&\qquad\quad \left. - \frac{dr_0^2(A-3B)}{1-r_0} + \frac{dr_0^3B}{(1-r_0)^2}\right) \\
&\qquad \times \left(c+1+2(c-b)r_0 - \frac{c}{(1-r_0)^2} - \frac{dr_0^2}{1-r_0}\right)^{-1}.
\end{aligned} \tag{29}$$

Then, (26) yields

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \left|A - B\frac{zf_0'(z)}{f_0(z)}\right| \quad (-1 \le B < A \le 1,\ z = r_0), \tag{30}$$

or equivalently $f_0 \in \mathcal{ST}[A, B]$. ☐

Theorem 7. *Let $-1 \le B < A \le 1$. The $\mathcal{ST}[A, B]$-radius for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \le c/n$, $n \ge 3$ and $c > 0$, is the real root in $(0, 1)$ of the equation*

$$\begin{aligned}
&\left((c+1)(A-B) - (2(1-B)-(1-A))r\left(2b - \frac{c}{2}\right)\right) \\
&\quad \times (1-r) \\
&\quad = c(1-B) + c(1-A)(1-r)\frac{\log(1-r)}{r}.
\end{aligned} \tag{31}$$

This radius is sharp.

Proof. By Lemma 5, condition (27) assures that r_0 is the $\mathcal{ST}[A, B]$-radius of $f \in \mathcal{A}_b$ where r_0 is the real root of (31). Therefore, using (8) and (19) for $f \in \mathcal{A}_b$ yields

$$\begin{aligned}
&\sum_{n=2}^{\infty} ((1-B)n - (1-A))|a_n| r_0^{n-1} \\
&\quad \le 2(2(1-B)-(1-A))br_0 \\
&\qquad + \sum_{n=3}^{\infty} ((1-B)n - (1-A))\left(\frac{c}{n}\right) r_0^{n-1} \\
&\quad = 2(2(1-B)-(1-A))br_0 \\
&\qquad + c(1-B)\left(\frac{1}{1-r_0} - 1 - r_0\right) \\
&\qquad - c(1-A)\left(-\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2}\right) \\
&\quad = c(B-A) + (2(1-B)-(1-A))r_0\left(2b - \frac{c}{2}\right) \\
&\qquad + \frac{c(1-B)r_0 + c(1-A)(1-r_0)\log(1-r_0)}{(1-r_0)r_0} \\
&\quad = A - B.
\end{aligned} \tag{32}$$

The result is sharp for the function f_0 given by (21). Indeed, f_0 satisfies

$$\begin{aligned}
&\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| \\
&\quad = \frac{-(c/2-2b)r_0 + c/(1-r_0) + (c\log(1-r_0))/r_0}{(1+c) + (c/2-2b)r_0 + (c\log(1-r_0))/r_0}, \\
&\left|A - B\frac{zf_0'(z)}{f_0(z)}\right| \\
&\quad = \left((1+c)(A-B) + (A-2B)\left(\frac{c}{2} - 2b\right)r_0\right. \\
&\qquad \left. + \frac{cB}{1-r_0} + \frac{cA\log(1-r_0)}{r_0}\right) \\
&\qquad \times \left((1+c) + \left(\frac{c}{2} - 2b\right)r_0 + \frac{c\log(1-r_0)}{r_0}\right)^{-1},
\end{aligned} \tag{33}$$

at the root $z = r_0$ in $(0, 1)$ of (31). Evidently, the function f_0 satisfies (30), and hence the result is sharp. ☐

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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